EE817/IS 893

Cryptography Engineering and Cryptocurrency

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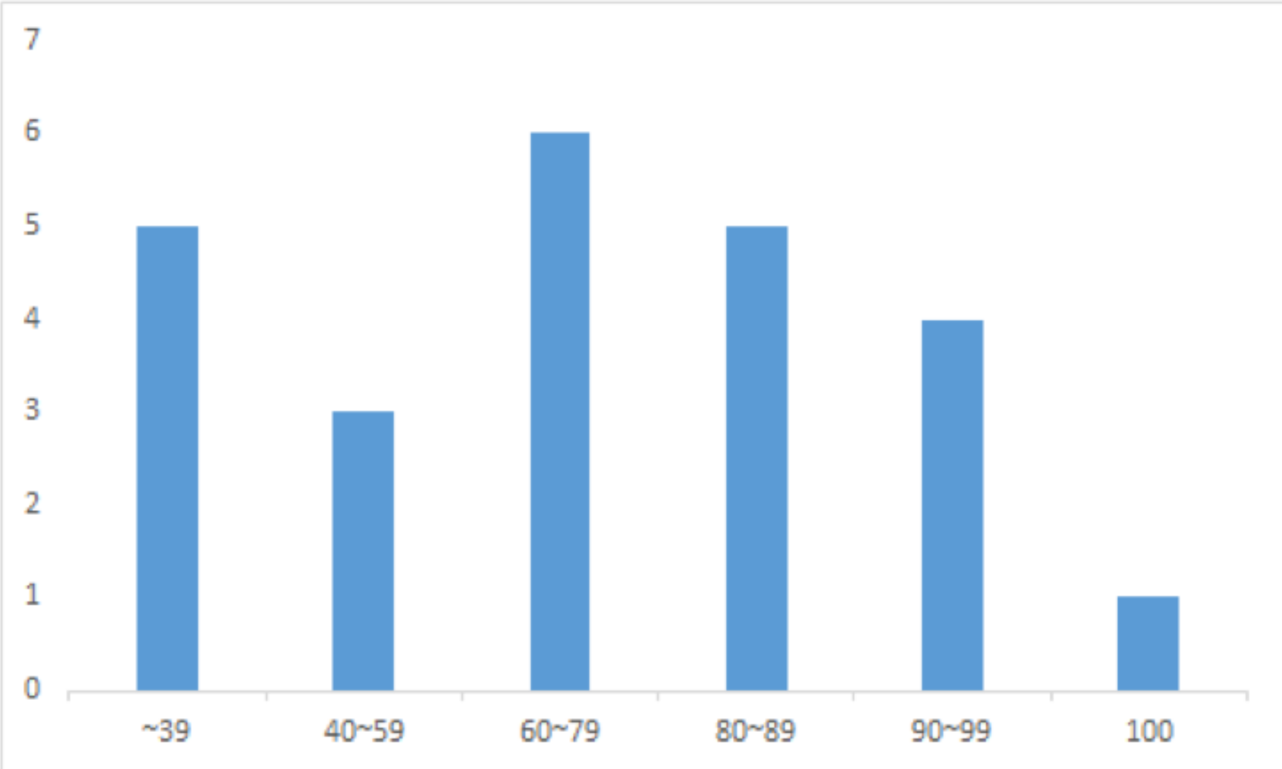
Admin Stuff

- Instructor Office Hours
  - And by appointment
- Pre-class Evaluation Test



Math, Math, Math!

Divisibility

- $\mathbb{Z} = \{\cdots -2, -1, 0, 1, 2, \cdots\}$
- Let a, b be integers. Then a ***divides*** b $(a|b)$
  - if $\exists$ c such that $b = ac$.
  - $16 \mid 32$? $16 \mid 0$?

Proof Techniques

- $p \Longrightarrow q$
  - When is this true?
  - How do you prove this?
  - What is this equivalent to?
  - Direct Proof
    - Show that the square of an even number is an even number
      - Rephrased: if n is even, then n^2 is even
    - Proof: Assume n is even
      - $\Longrightarrow$ Thus, $n = 2k$, for some k (definition of even numbers)
      - $\Longrightarrow$ $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$
      - $\Longrightarrow$ As n^2 is 2 times an integer, n^2 is thus even
  - Indirect Proof (contrapositive)
    - If n^2 is an odd integer then n is an odd integer
      - This is equivalent to: if n is even, then n^2 is even

Proof Techniques

- If n is an integer and n^3+5 is odd, then n is even
  - Which one do we need to use?

- Proof by contradiction
  - Theorem (by Euclid): There are infinitely many prime numbers.
- Proof by cases
  - Prove that $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$ for all integer n.
- Existence Proof: $\exists x\ P(x)$
  - constructive: Find a specific value of c for which P(c) is true
    - a square exists that is the sum of two other squares.
  - Nonconstructive: Show that such a c exists, but don't actually find it
    - We will see examples.

Proof Techniques

- Universal Proof: $\forall$ x P(x)

- Uniqueness Proof
  - If the real number equation 5x+3=a has a solution then it is unique

- Induction
  - Quiz

- Prove or disprove that $n^2-79n+1601$ is a prime whenever n is a positive integer

Forwards vs. Backwards reasoning

- Example: Prove that $(a+b)/2 > \sqrt{(ab)}$ when $a \neq b$, $a>0$, and $b>0$

(Pf) $(a - b)^2 > 0$

➔ $a^2 + 2ab + b^2 - 4ab > 0$

➔ $(a+b)^2 > 4ab$

➔ $((a+b)/2)^2 > ab$

➔ $(a+b)/2 > \sqrt{(ab)}$

Divisibility

- Let a, b, c be integers.
  - $a|a$

    We need to find c such that $a = ac$.

    $c = 1$.
  - if $a|b$ and $b|c$, then $a|c$

    Assume $a \mid b$ and $b \mid c$.

    $\Longrightarrow \exists$ integers k_1, k_2 such that $b = k_1 a$ and $c = k_2 b$

    $\Longrightarrow c = k_1 k_2\, a$. Since $k_1 \cdot k_2$ is an integer, $a \mid c$.

    » Which proof technique we used?
  - if $a|b$ and $a|c$, then $a|(bx+cy)$ for all $x, y \in \mathbb{Z}$
  - if $a|b$ and $b|a$, then $a = \pm b$

Quotient and remainder

- Let a, b be integers and a>0. Then, there exist unique integers q and r such that

$$b = a q + r, \qquad 0 \leq r < a.$$

Proof) Assume that $b \geq 0$. It is clear that $\exists$ n such that $n a > b$. Let $q + 1$ be the least such n. Then $(q+1) a > b \geq q a$.

Let $r=b-qa$. Then, $b \geq qa$ implies $r=b-qa \geq 0$. Finally $(q+1)a=qa+a>b$ implies that $r=b-qa<a$.

To show the uniqueness, suppose $\exists$ q_1 and r_1 such that $b=qa+r=q_1a+r_1$, $0 \leq r, r_1 < a$. Assume $r \geq r_1$. Then $0 \geq r - r_1 < a$, and $(q-q_1)a=r-r_1$. Then $a|r-r_1$. If $r-r_1>0$, $a \leq r-r_1$ (since $a|r-r_1$). (*) Therefore, $r=r_1$. Then $q = q_1$.

Exercise

- If a, b, c are nonzero integers, prove that $ac \mid bc$ if and only if $a \mid b$.

- Show that for any integer n, n^2 cannot be of the form $3k + 2$.

GCD, LCM

- c is a **common divisor** of a and b if c|a and c|b
- **d = gcd(a,b)** is the largest positive integer that divides both a and b, more formally
  - d > 0
  - d | a and d | b
  - e | a and e | b implies e | d
- **lcm(a,b)** is the smallest positive integer divisible by both a and b
- lcm(a,b)=a*b/gcd(a,b)
- a and b are said to be ***relatively prime*** or ***coprime*** if gcd(a,b)=1

Existence of GCD

- Let a and b be integers (a or b is not zero). Then $d = \gcd(a, b)$ exist.
- Proof (non-constrctive proof)

Let $S = \{ax + by \mid x, y \in \mathbb{Z}\}$. Let d be the least positive integer in S. Then $d = ax_0 + by_0$.

claim: $d = \gcd(a, b)$

i) $d>0$

iii) $e|a$ and $e|b$, then $e|d$.

ii) $d|a$, $d|b$

Let $a = dq+r$, $0 \le r < d$. Then $r = a - qd = a - q(ax_0+by_0) = a(1-qx_0) - qby_0$. clearly $r \in S$. And $r < d$. Since d is the least positive integer in S, $r = 0$. Therefore, $a = dq$.

- Proof (constructive proof) next page!

Existence of GCD (cnt.)

- constructive proof (Extended Eucledean Algorithm)

$b = q_1 a + r_1, \quad 0 < r_1 < a$

$a = q_2 r_1 + r_2, \quad 0 < r_2 < r_1$

$r_1 = q_3 r_2 + r_3, \quad 0 < r_3 < r_2$

...

$r_{n-2} = q_n r_{n-1} + r_n, \quad 0 < r_n < r_{n-1}$

$r_{n-1} = q_{n+1} r_n$, (no remainder)

Since the remainder decreases and it is an integer, it will be 0 eventually.

claim) $r_n = \gcd(a, b)$

i) $r_n > 0$

ii) $r_n \mid a$, $r_n \mid b$

iii) $e \mid a$, $e \mid b \Rightarrow e \mid r_n$.

Example

$51329 = 21\ 2437 + 152$

$2437 = 16\ 152 + 5$

$152 = 30\ 5 + 2$

$5 = 2\ 2 + 1$

$2 = 2\ 1 + 0$

$1 = 5 - 2\ 2$

$= 5 - 2\ (152 - 30 * 5)$

$= -2\ 152 + 61\ 5$

$= -2\ 152 + 61\ (2437 - 16\ 152)$

$= 61\ 2437 - 978\ 152$

$= 61\ 2437 - 978\ (51329 - 21\ 2437)$

$= -978\ 51329 + 20599\ 2437$

Summary

- $d = \gcd(a, b) \Rightarrow \exists\ x, y$ such that $d = a x + b y$.
- $\gcd(a, 0) = ?$
- **Euclidean Algorithm** to compute GCD
  - Input: a, b with $a \geq b$
  - Output: gcd (a, b)
  - Algorithm
    - » while $b \neq 0$
      - Set $r \leftarrow a \bmod b$, $a \leftarrow b$, $b \leftarrow r$
    - » return (a)
  - complexity?

A Few more useful stuffs

- Let $d = \gcd(a, b)$
  - $\gcd(a/d, b/d) = ?$
  - $a \mid bc$ and $d = 1 \Longrightarrow ?$
  - $a \mid bc \Longrightarrow (a/d) \mid c$
  - $\gcd(ma, mb) = md$ if $m > 0$
- $\gcd(n, n+1)$?
- $\gcd(a, b) = \gcd(a + kb, b)$?

Prime

- $p \geq 2$ is prime if
  - $a \mid p \Rightarrow a = \pm 1$ or $\pm p$
  - Hereafter, p is prime
- [Euclid] $p \mid ab \Rightarrow p \mid a$ or $p \mid b$
- [Euclid] There are infinite number of primes.
- *Prime number theorem:*
  - let $\pi(x)$ denote the number of prime numbers $\leq x$, then

    $\lim_{x \to \infty} \pi(x)/(x/\ln x) = 1$
- **Euler phi function:** For $n \geq 1$, let $f(n)$ denote the number of integers in [1, n] which are relatively prime to n.
  - if p is a prime then $f(p)=p-1$
  - if p is a prime, then $\phi(p^r) = p^{r-1}(p-1)$.
  - f is multiplicative. That is if $\gcd(m,n)=1$ then $f(m*n) = f(n) * f(m)$

Fundamental theorem of arithmetic

- Every positive integer greater than 1 can be uniquely written as a prime or as the product of two or more primes where the prime factors are written in order of non-decreasing size

- Examples
  - 100 = 2 * 2 * 5 * 5
  - 182 = 2 * 7 * 13
  - 29820 = 2 * 2 * 3 * 5 * 7 * 71

Pairwise relative prime

- A set of integers $a_1, a_2, \cdots a_n$ are pairwise relatively prime if, for all pairs of numbers, they are relatively prime
  - Formally: The integers $a_1, a_2, \cdots a_n$ are pairwise relatively prime if $\gcd(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n$.
- Example: are 10, 17, and 21 pairwise relatively prime?
  - gcd(10,17) = 1, gcd (17, 21) = 1, and gcd (21, 10) = 1
  - Thus, they are pairwise relatively prime
- Example: are 10, 19, and 24 pairwise relatively prime?
  - Since gcd(10,24) ≠ 1, they are not

Modular arithmetic

- If a and b are integers and m is a positive integer, then a is *congruent to* b *modulo* m if m divides $a-b$
  - Notation: $a \equiv b \pmod{m}$
  - Rephrased: $m \mid a-b$
  - Rephrased: $a \bmod m = b$
  - If they are not congruent: $a \not\equiv b \pmod{m}$

- Example: Is 17 congruent to 5 modulo 6?
  - Rephrased: $17 \equiv 5 \pmod{6}$
  - As 6 divides 17-5, they are congruent
- Example: Is 24 congruent to 14 modulo 6?
  - Rephrased: $24 \equiv 14 \pmod{6}$
  - As 6 does not divide 24-14 = 10, they are not congruent

Example (World of mod n)

...	-2n	-n	0	n	2n	3n	4n	...	0
...	-2n+1	-n+1	1	n+1	2n+1	3n+1	4n+1	...	1
...	-2n+2	-n+2	2	n+2	2n+2	3n+2	4n+2	...	2
...	...	...	...	...	...	...	...	...	...
...	-n-1	-1	n-1	2n-1	3n-1	4n-1	5n-1	...	n-1

More on congruence

- Every integer is either of the form 4k, 4k+1, 4k+2, 4k+3.
- Every integer is either of the form 0 mod 4, 1 mod 4, 2 mod 4, 3 mod 4
- $y^2 - x^2 - 2 \equiv 0$ mod 4 has no solution.

- Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if a **mod** $m = b$ **mod** m
- Example: Is 17 congruent to 5 modulo 6?
  - Rephrased: does $17 \equiv 5 \pmod{6}$?
  - 17 mod 6 = 5 mod 6
- Example: Is 24 congruent to 14 modulo 6?
  - Rephrased: $24 \equiv 14 \pmod{6}$
  - 24 mod 6 $\neq$ 14 mod 6

Even more on congruence

- Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$

- Example: 17 and 5 are congruent modulo 6
  - 17 = 5 + 2*6
  - 5 = 17 -2*6

- Let a, b, c be integers.
  - $a \equiv a \bmod n$
  - $a \equiv b \bmod n \Longrightarrow b \equiv a \bmod n$
  - $a \equiv b \bmod n$ and $b \equiv c \bmod n \Longrightarrow a \equiv c \bmod n$.

Even even more on congruence

- Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a+c \equiv (b+d) \pmod{m}$ and $ac \equiv bd \pmod{m}$

- Example
  - We know that $7 \equiv 2 \pmod{5}$ and $11 \equiv 1 \pmod{5}$
  - Thus, $7+11 \equiv (2+1) \pmod{5}$, or $18 \equiv 3 \pmod{5}$
  - Thus, $7*11 \equiv 2*1 \pmod{5}$, or $77 \equiv 2 \pmod{5}$

- An integer x is congruent to one and only one of the integers 0, 1, 2, ⋯, n-1 mod n.

The caesar cipher

- Julius caesar used this to encrypt messages

- A function f to encrypt a letter is defined as:
  $f(p) = (p+3) \bmod 26$
  - Where p is a letter (0 is A, 1 is B, 25 is Z, etc.)

- Decryption: $f^{-1}(p) = (p-3) \bmod 26$

- This is called a substitution cipher
  - You are substituting one letter with another

Arithmetic Inverse

- Let a be an integer. a* is an arithmetic inverse of a modulo n, if $a\,a^* \equiv 1 \bmod n$.

- Suppose that gcd(a, n) =1. Then a has an arithmetic inverse modulo n.

- Suppose gcd(a, n) = 1.

  Then $ax \equiv ay \bmod n \Longrightarrow x \equiv y \bmod n$.

- $x^2+1 \equiv 0 \bmod 8$ has no solution.

Equations

- $2x \equiv 5 \bmod 3$

  $\Longrightarrow 2x \equiv 2 \bmod 3$

  $\Longrightarrow 2^* \, 2x \equiv 2^* \, 2 \bmod 3$

  $\Longrightarrow x \equiv 1 \bmod 3 \quad (2^* \equiv 2 \bmod 3)$

- $3x \equiv 7 \bmod 5$

  $\Longrightarrow 3x \equiv 2 \bmod 5$

  $\Longrightarrow 3^* \, 3x \equiv 3^* \, 2 \bmod 5$

  $\Longrightarrow x \equiv 4 \bmod 5 \quad (3^* \equiv 2 \bmod 5)$

Summary on congruence

- Notation: $a \equiv b \pmod{m}$
  - Rephrased: $m \mid a-b$
  - Rephrased: $a \bmod m = b$
  - Rephrased: $a = b + mk$, for some integer k,
- Every integer is either of the form
  - 4k, 4k+1, 4k+2, or 4k+3.
  - 0 mod 4, 1 mod 4, 2 mod 4, or 3 mod 4
- If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then
  - $a+c \equiv (b+d) \pmod{m}$
  - $ac \equiv bd \pmod{m}$
- Suppose that gcd(a, n) =1. Then a has an arithmetic inverse a^* modulo n, i.e. a $a^* \equiv a^* a \equiv 1 \bmod n$.

Cute Exercise

- A number is divisible by 3, if sum of the all digits is divisible by 3. Why does this work?

Z_n, Z_n^*

- $Z_n = \{0, 1, 2, 3, \cdots, n-1\}$
- $Z_n^* = \{x \mid x \in Z_n \text{ and } \gcd(x, n) = 1\}$.

- $Z_6 = \{0, 1, 2, 3, 4, 5\}$
- $Z_6^* = \{1, 5\}$

- For a set S, |S| means the number of element in S.

- $|Z_n| = n$
- $|Z_n^*| = \phi(n)$

cardinality

- For finite (only) sets, cardinality is the number of elements in the set

- For finite and infinite sets, two sets A and B have the same cardinality if there is a one-to-one correspondence from A to B

Counting

Multiplication rule

- If there are n_1 ways to do task1, and n_2 ways to do task2
  - Then there are $n_1 n_2$ ways to do both tasks in sequence.
- Example
  - There are 18 math majors and 325 CS majors
  - How many ways are there to pick one math major and one CS major?

Addition rule

- If there are n_1 ways to do task1, and n_2 ways to do task2
  - If these tasks can be done at the same time, then…
  - Then there are n_1+n_2 ways to do one of the two tasks
- How many ways are there to pick one math major or one CS major?

The inclusion-exclusion principle

- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Permutation, combination

- An *r*-permutation is an ordered arrangement of *r* elements of the set: $P(n, r)$, ${}_nP_r$
  - How many poker hands (with ordering)?
  - $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$
    $= n! / (n-r)!$
- combination: when order does not matter…
  - In poker, the following two hands are equivalent:
    - A♦, 5♥, 7♣, 10♠, K♠
    - K♠, 10♠, 7♣, 5♥, A♦
  - The number of *r*-combinations of a set with *n* elements, where *n* is non-negative and $0 \le r \le n$ is:
    $c(n, r) = n! / (r! (n-r)!)$
  - $(x+y)^n$

Probability definition

- The probability of an event occurring is:

  $P(E) = |E| / |S|$

  - Where E is the set of desired events (outcomes)
  - Where S is the set of all possible events (outcomes)
  - Note that $0 \leq |E| \leq |S|$
    - Thus, the probability will always between 0 and 1
    - An event that will never happen has probability 0
    - An event that will always happen has probability 1

What's behind door number three?

- The Monty Hall problem paradox
  - consider a game show where a prize (a car) is behind one of three doors
  - The other two doors do not have prizes (goats instead)
  - After picking one of the doors, the host (Monty Hall) opens a different door to show you that the door he opened is not the prize
  - Do you change your decision?
- Your initial probability to win (i.e. pick the right door) is 1/3
- What is your chance of winning if you change your choice after Monty opens a wrong door?
- After Monty opens a wrong door, if you change your choice, your chance of winning is 2/3
  - Thus, your chance of winning doubles if you change
  - Huh?

Assigning Probability

- S: Sample space
- p(s): probability that s happens.
  - $0 \leq p(s) \leq 1$ for each $s \in S$
  - $\sum_{s \in S} p(s) = 1$
- The function p is called probability distribution
- Example
  - Fair coin: p(H) = 1/2, p(T) = 1/2
  - Biased coin where heads comes up twice as often as tail
    - p(H) = 2 p(T)
    - p(H) + p(T) = 1 $\Rightarrow$ 3 p(T) = 1 $\Rightarrow$ p(T) = 1/3, p(H) = 2/3

More…

- Uniform distribution
  - Each element $s \in S$ ($|S| = n$) is assigned with the probability $1/n$.
- Random
  - The experiment of selecting an element from a sample space with uniform distribution.
- Probability of the event E
  - $p(E) = \sum_{s \in E} p(s)$.
- Example
  - A die is biased so that 3 appears twice as often as others
    - $p(1) = p(2) = p(4) = p(5) = p(6) = 1/7$, $p(3) = 2/7$
  - $p(O)$ where O is the event that an odd number appears
    - $p(O) = p(1) + p(3) + p(5) = 4/7$.

Combination of Events

- Still
  - $P(E^c) = 1 - P(E)$
  - $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$
    - $E_1 \cap E_2 = \emptyset \Rightarrow P(E_1 \cup E_2) = P(E_1) + P(E_2)$
    - For all $i \neq j$, $E_i \cap E_i = \emptyset \Rightarrow P(\cup_i E_i) = \sum_i P(E_i)$

Conditional Probability

- Flip coin 3 times
  - all eight possibility are equally likely.
  - Suppose we know that the first coin was tail (Event F). What is the probability that we have odd number of tails (Event E)?
    - only four cases: TTT, TTH, THT, THH
    - So 2/4 = 1/2.

- Conditional probability of E given F
  - We need to use F as the sample space
  - For the outcome of E to occur, the outcome must belong to $E \cap F$.
  - $p(E \mid F) = p(E \cap F) / p(F)$.

Bernoulli Trials & Binomial Distribution

- Beronoulli trial
  - an experiment with only two possible outcomes
  - i.e. 0 (failure) and 1 (success).
  - If p is the probability of success and q is the probability of failure, $p + q = 1$.
- A biased coin with probability of heads 2/3
  - What is the probability that four heads up out of 7 trials?

Random variable

- A random variable is a function from the sample space of an experiment to the set of real numbers.
  - Random variable assigns a real number to each possible outcome.
  - Random variable is not variable! not random!
- Example: three times coin flipping
  - Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome
  - $X(HHH) = 3$, $X(THH) = X(HTH) = X(HHT) = 2$, $X(TTH) = X(THT) = X(HTT) = 1$, $X(TTT) = 0$
- The distribution of a random variable X on a sample space S is the set of pairs $(r, p(X=r))$ for all $r \in X(S)$
  - where $p(X=r)$ is the probability that X takes value r.
  - $p(X=3) = 1/8$, $p(X=2) = 3/8$, $p(X=1) = 3/8$, $p(X=0) = 1/8$

Expected value

- The expected value of the random variable $X(s)$ on the sample space S is equal to

  $E(X) = \sum_{s \in S} p(s)\, X(s)$

- Expected value of a Die
  - X is the number that comes up when a die is rolled.
  - What is the expected value of X?
  - $E(X) = 1/6\ 1 + 1/6\ 2 + 1/6\ 3 + \cdots 1/6\ 6 = 21/6 = 7/2$
- Three times coin flipping example
  - X: number of heads
  - $E(X) = 1/8\ 3 + 3/8\ 2 + 3/8\ 1 + 1/8\ 0 = 12/8 = 3/2$

Questions?

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